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An application of the Romberg Extrapolation on Modified Trapezoidal Quadrature method for solving linear Volterra Integral Equations of the second type

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ABSTRACT: In this article we have used modified trapezoidal quadrature method to solve the Volterra integral equation (VIE) of the second type. Using Romberg extrapolation, the speed and accuracy of the approximations obtained for the integral equation will be improved. The calculated results are compared to results from the trapezoidal quadrature method.

Keywords: Linear Volterra integral equations of the second kind (VIE), Modified trapezoidal quadrature method, Romberg extrapolation.

INTRODUCTION

The Volterra integral equations are used in many problems of physics such as, the particle transport problems of astrophysics, potential theory and Dirichle problem, electrostatic and radiative heat transfer problems and in some engineering fields (Wazwaz, 1996, Wang, 2006; Saberi-Nadjafi and Heidari, 2006). Integral equation theory and its application in applied mathematics is an important issue. There are several numerical approximations to the results of linear Volterra integral equations of the second type. Many different basic functions have been recently utilized to estimate the result of integral equations, such as orthogonal bases and wavelets (Jung and Schanfelberger, 1992, Maleknejad and Hadizadeh, 1999, Razzaghi and Arabshahi, 1989). Here we use modified the trapezoidal quadrature rule to solve the linear Volterra integral equations of the second type. The general form of the linear VIE of the second type is as follows:

$$u(x)=f(x)+\int_a^x k(x,t)u(t)dt \qquad a\le x\le b \tag{1}$$

The kernel function of the integral equation, $k(x,t)$, and the function f(x) are known and u(x) is the unknown function to be determined using the modified trapezoidal quadrature method.

Quadrature Method

Different quadrature methods could be used to solve linear VIEs of the second type. The modified trapezoid quadrature method has been used here for the numerical computations. We consider the given integral equation (1) and we put

$$\begin{cases} x_0=a \\ x_i=x_{i-1}+h=x_0+ih \end{cases} \tag{2}$$

For which the unknown function values at any mesh point are described as

$$\begin{cases} u(x_0)=f(x_0) \\ u(x_i)=f(x_i)+\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j} k(x_i,t)u(t)dt \end{cases} \qquad i=1,2,...,n \tag{3}$$

The approximation of integrals in (3) using the modified trapezoidal rule will result in the following system of equations:

$$\begin{cases} u_0 = f_0 \\ u_1 = f_1 + \frac{h}{2}(k_{1,0}u_0 + k_{1,1}u_1) + \frac{h^2}{12}(J_{1,0}u_0 + k_{1,0}u_0') - \frac{h^2}{12}(J_{1,1}u_1 + k_{1,1}u_1') \\ \vdots \\ u_i = f_i + \frac{h}{2}(k_{i0}u_0 + 2\sum_{j=1}^{i-1} k_{ij}u_j + k_{ii}u_i) + \frac{h^2}{12}(J_{i0}u_0 + k_{i0}u_0') - \frac{h^2}{12}(J_{i,i}u_i + k_{ii}u_i') \\ \vdots \\ u_n = f_n + \frac{h}{2}(k_{n0}u_0 + 2\sum_{j=1}^{i-1} k_{ij}u_j + k_{nn}u_n) + \frac{h^2}{12}(J_{n0}u_0 + k_{n0}u_0') - \frac{h^2}{12}(J_{nn}u_n + k_{nn}u_n') \end{cases} \quad (4)$$

In the above relations $J(x,t) = \frac{\partial k(x,t)}{\partial t}$, and $u_0', u_1', \ldots, u_n'$ can be approximated using the modified trapezoidal and trapezoidal rules. For this purpose the derivative of the integral equation (1) with respect to x is:

$$u'(x) = f'(x) + k(x,x)u(x) + \int_0^x \frac{\partial k(x,t)}{\partial x} u(t)\,dt \quad (5)$$

Supposing $m(x,t) = \frac{\partial \mathrm{k(x,t)}}{\partial \mathrm{x}}$, then (5) is as follows:

$$u'(x) = f'(x) + k(x,x)u(x) + \int_0^x m(x,t)u(t)\,dt \quad (6)$$

As mentioned to approximate $u_0', u_1', \ldots, u_n'$ two cases are considered:

***First case**:*

$u_0', u_1', \ldots, u_n'$ can be approximated using the simple trapezoid method.

***Second case**:*

$u_0', u_1', \ldots, u_n'$ can be approximated using the modified trapezoid method.

If we consider $l(x,t) = \frac{\partial^2 \mathrm{k(x,t)}}{\partial \mathrm{t} \partial \mathrm{x}}$.

In the first case, equation (6) is solved using the simple trapezoidal integration method, in which case the approximation $u_i', i = 0,1,\ldots,n$ are obtained as follows:

$$\begin{cases} u_0' = f_0' + k_{0,0}f_0 \\ \vdots \\ u_i' = f_i' + k_{i,i}u_i + \frac{h}{2}(m_{i,0}u_0 + 2\sum_{j=1}^{i-1} m_{i,j}u_j + m_{i,i}u_i) \\ \vdots \\ u_n' = f_n' + k_{n,n}u_n + \frac{h}{2}(m_{n,0}u_0 + 2\sum_{j=1}^{i-1} m_{i,j}u_j + m_{n,n}u_n) \end{cases} \quad (7)$$

Substituting (7) in (4), $u_i, i = 0,1,\ldots,n$ are obtained as:

$$\begin{cases} u_0 = f_0 \\ \vdots \\ u_i = \dfrac{f_i + \frac{h}{2}(k_{i,0}u_0 + 2\sum_{j=1}^{i-1} k_{i,j}u_j) - \frac{h^3}{24}(k_{i,i}\, m_{i,0}u_0 + 2k_{i,i}\sum_{j=1}^{i-1} m_{i,j}u_j)}{[1 - \frac{h}{2}k_{i,i} + \frac{h^2}{12}J_{i,i} + \frac{h^2}{12}k_{i,i}^2 + \frac{h^3}{24}k_{i,i}\, m_{i,i}]} \\ \quad + \dfrac{\frac{h^2}{12}(k_{i,0}f_0' + k_{i,0}\, k_{0,0} f_0 - k_{i,i} f_i' + J_{i,0}u_0)}{[1 - \frac{h}{2}k_{i,i} + \frac{h^2}{12}J_{i,i} + \frac{h^2}{12}k_{i,i}^2 + \frac{h^3}{24}k_{i,i}\, m_{i,i}]} \\ \vdots \\ u_n = \dfrac{f_n + \frac{h}{2}(k_{n,0}u_0 + 2\sum_{j=1}^{n-1} k_{i,j}u_j) - \frac{h^3}{24}(k_{n,n}\, m_{n,0}u_0 + 2k_{n,n}\sum_{j=1}^{n-1} m_{i,j}u_j)}{[1 - \frac{h}{2}k_{n,n} + \frac{h^2}{12}J_{n,n} + \frac{h^2}{12}k_{n,n}^2 + \frac{h^3}{24}k_{n,n}\, m_{n,n}]} \\ \quad + \dfrac{\frac{h^2}{12}(k_{n,0}f_0' + k_{n,0}\, k_{0,0} f_0 - k_{n,n} f_n' + J_{n,0}u_0)}{[1 - \frac{h}{2}k_{n,n} + \frac{h^2}{12}J_{n,n} + \frac{h^2}{12}k_{n,n}^2 + \frac{h^3}{24}k_{n,n}\, m_{n,n}]} \end{cases} \qquad (8)$$

In the second case, equation (6) is solved using the modified trapezoidal integration method, in which case the resulting $u_i', i = 0,1,...,n$ are:

$$\begin{cases} u_0' = f_0' + k_{0,0} f_0 \\ \vdots \\ u_i' = f_i' + k_{i,i}u_i + \frac{h}{2}(m_{i0}u_0 + 2\sum_{j=1}^{i-1} m_{i,j}u_j + m_{i,i}u_i) + \frac{h^2}{12}(l_{i,0}u_0 + m_{i,0}u_0') - \frac{h^2}{12}(l_{i,i}u_i + m_{i,i}u_i') \\ \vdots \\ u_n' = f_n' + k_{n,n}u_n + \frac{h}{2}(m_{n,0}u_0 + 2\sum_{j=1}^{i-1} m_{i,j}u_j + m_{n,n}u_n) + \frac{h^2}{12}(l_{n,0}u_0 + m_{n,0}u_0') - \frac{h^2}{12}(l_{n,n}u_n + m_{n,n}u_n') \end{cases} \qquad (9)$$

Using the simple mathematical rules, approximations $u_i' \;\; i = 0,1,...,n$ are obtained as bellow:

$$\begin{cases} u_0' = f_0' + k_{i,i} f_0 \\ \vdots \\ u_i' = \dfrac{f_i' + k_{i,i}u_i + \frac{h}{2}(m_{i,0}u_0 + m_{i,i}u_i + 2\sum_{j=1}^{i-1} m_{i,j}u_j)}{1 + \frac{h^2}{12}m_{i,i}} \\ \quad + \dfrac{\frac{h^2}{12}(l_{i,0}u_0 + m_{i,0}f_0' + m_{i,0}\, k_{0,0} f_0 - l_{i,i}u_i)}{1 + \frac{h^2}{12}m_{i,i}} \\ \vdots \\ u_n' = \dfrac{f_n' + k_{n,n}u_n + \frac{h}{2}(m_{n,0}u_0 + m_{n,n}u_n + 2\sum_{j=1}^{i-1} m_{i,j}u_j)}{1 + \frac{h^2}{12}m_{n,n}} \\ \quad + \dfrac{\frac{h^2}{12}(l_{n,0}u_0 + m_{n,0}f_0' + m_{n,0}\, k_{0,0} f_0 - l_{n,n}u_n)}{1 + \frac{h^2}{12}m_{n,n}} \end{cases} \qquad (10)$$

Substituting these obtained approximations in (4), $u_i \;\; i=0,1,...,n$ are obtained as follows:

$$\begin{cases} u_0 = f_0 \\ \vdots \\ u_i = \dfrac{f_i + \frac{h}{2}(k_{i0}u_0 + 2h\sum_{j=1}^{i-1} k_{i,j}u_j) + \frac{h^2}{12}(J_{i,0}u_0 + k_{i,0}f_0' + k_{i,0}k_{0,0}f_0 - \frac{k_{i,i}f_i'}{1+\frac{h^2}{12}m_{i,i}})}{1 - \frac{h}{2}k_{i,i} + \frac{h^2}{12}J_{i,i} + \frac{\frac{h^2}{12}}{1+\frac{h^2}{12}m_{i,i}}(k_{i,i} + \frac{h}{2}k_{i,i}\,m_{i,i} - \frac{h^2}{12}k_{i,i}\,l_{i,i})} \\ \quad - \dfrac{\frac{\frac{h^4}{144}}{1+\frac{h^2}{12}m_{i,i}}(k_{i,i}\,l_{i,0}u_0 + k_{i,i}\,m_{i,0}f_0' + k_{i,i}\,m_{i,0}k_{0,0}f_0)}{1 - \frac{h}{2}k_{i,i} + \frac{h^2}{12}J_{i,i} + \frac{\frac{h^2}{12}}{1+\frac{h^2}{12}m_{i,i}}(k_{i,i} + \frac{h}{2}k_{i,i}\,m_{i,i} - \frac{h^2}{12}k_{i,i}\,l_{i,i})} \\ \vdots \\ u_n = \dfrac{f_n + \frac{h}{2}(k_{n0}u_0 + 2\sum_{j=1}^{n-1} k_{n,j}u_j) + \frac{h^2}{12}(J_{n0}u_0 + k_{n0}f_0' + k_{n,0}k_{0,0}f_0 - \frac{k_{n,n}f_n'}{1+\frac{h^2}{12}m_{n,n}})}{1 - \frac{h}{2}k_{n,n} + \frac{h^2}{12}J_{n,n} + \frac{\frac{h^2}{12}}{1+\frac{h^2}{12}m_{n,n}}(k_{n,n} + \frac{h}{2}k_{n,n}m_{n,n} - \frac{h^2}{12}k_{n,n}\,l_{n,n})} \\ \quad - \dfrac{\frac{\frac{h^4}{144}}{1+\frac{h^2}{12}m_{n,n}}(k_{n,n}\,l_{n,0}u_0 + k_{n,n}\,m_{n,0}f' + k_{n,n}\,m_{n,0}k_{0,0}f_0)}{1 - \frac{h}{2}k_{n,n} + \frac{h^2}{12}J_{n,n} + \frac{\frac{h^2}{12}}{1+\frac{h^2}{12}m_{n,n}}(k_{n,n} + \frac{h}{2}k_{n,n}\,m_{n,n} - \frac{h^2}{12}k_{n,n}\,l_{n,n})} \\ \quad - \dfrac{\frac{\frac{h^3}{24}}{1+\frac{h^2}{12}m_{n,n}}(k_{n,n}\,m_{n,0}u_0 + 2k_{n,n}\sum_{j=1}^{n-1} m_{n,j}u_j)}{1 - \frac{h}{2}k_{n,n} + \frac{h^2}{12}J_{n,n} + \frac{\frac{h^2}{12}}{1+\frac{h^2}{12}m_{n,n}}(k_{n,n} + \frac{h}{2}k_{n,n}\,m_{n,n} - \frac{h^2}{12}k_{n,n}\,l_{n,n})} \end{cases} \tag{11}$$

Romberg Extrapolation

The integral **I** through numerical $T_m(h)$ method, with length step of h and the convergence of order k, is approximated by $I = T_m(h) + C(h)^k$. For the sake of a better approximation, the integral can be calculated with the length step of h/2 and the same order of convergence as $I = T_m\left(\frac{h}{2}\right) + C(\frac{h}{2})^k$.

To speed up the pace of the convergence in our methods and obtaining more accurate results, the Romberg extrapolation model is used.

With the intensive use of the Romberg numerical integration algorithm, now a method is applied bellow to improve the numerical solution of the given integral with the convergence order (k+2):

$$T_{m+1}(h) = \frac{4^k T_m(\frac{h}{2}) - T_m(h)}{4^k - 1} \tag{12}$$

Table 1 shows the application of the described algorithm on quadrature method for solving linear Volterra integral equations of the second type.

Table 1. The algorithm for Romberg extrapolation applied on quadrature method for solving linear Volterra integral equations of the second type

m=1		m=2		m=3		m=4		m=5
$T_1(h)$	$\rightarrow \nearrow$	$T_2(h)$	$\rightarrow \nearrow$	$T_3(h)$	$\rightarrow \nearrow$	$T_4(h)$	$\rightarrow \nearrow$	$T_5(h)$
$T_1(\frac{h}{2})$	$\rightarrow \nearrow$	$T_2(\frac{h}{2})$	$\rightarrow \nearrow$	$T_3(\frac{h}{2})$	$\rightarrow \nearrow$	$T_4(\frac{h}{2})$		
$T_1(\frac{h}{4})$	$\rightarrow \nearrow$	$T_2(\frac{h}{4})$	$\rightarrow \nearrow$	$T_3(\frac{h}{4})$				
$T_1(\frac{h}{8})$	$\rightarrow \nearrow$	$T_2(\frac{h}{8})$						
$T_1(\frac{h}{16})$								
k=1		k=2		k=3		k=4		

Numerical Exampl
Example 1
The presented algorithm has been applied for solving linear Volterra integral equation of the second type

$$u(x) = x + \frac{1}{5}\int_0^x x\,t\,u(t)dt \qquad 0 \le x \le 2 \qquad (13)$$

The exact solution of this equation is

$$u(x) = x\,e^{\frac{x^3}{15}} \qquad (14)$$

Tables 2 and 3 show absolute errors of numerical results calculated according to the presented algorithm, and for both cases described above. The results after Romberg extrapolation are more accurate than results obtained utilizing the quadrature formula over more points. For example, the calculated values $T_2(\frac{h}{2})$ and $T_3(h)$ are much more accurate than $T_1(\frac{h}{8})$.

Table 2. Absolute errors of the numerical solution of (13) at x = 2 with h = 0.25 in the first case

m=1	m=2	m=3	m=4	m=5
$2.23*10^{-5}$	$5.58*10^{-5}$	$1.03*10^{-8}$	$7.26*10^{-12}$	$8.88*10^{-16}$
$1.39*10^{-5}$	$3.48*10^{-6}$	$1.68*10^{-10}$	$2.75*10^{-14}$	
$8.69*10^{-7}$	$2.17*10^{-7}$	$2.65*10^{-12}$		
$5.43*10^{-8}$	$1.36*10^{-8}$			
$3.39*10^{-9}$				
k=1	k=2	k=3	k=4	

Table 3. Absolute errors of the numerical solution of (13) at x = 2 with h = 0.25 in the second case

m=1	m=2	m=3	m=4	m=5
$4.99*10^{-3}$	$5.05*10^{-5}$	$4.75*10^{-8}$	$5.64*10^{-10}$	$1.54*10^{-11}$
$1.21*10^{-3}$	$3.11*10^{-6}$	$1.30*10^{-9}$	$1.75*10^{-11}$	
$3.00*10^{-4}$	$1.93*10^{-7}$	$3.75*10^{-11}$		
$7.49*10^{-5}$	$1.20*10^{-8}$			
$1.87*10^{-5}$				
k=1	k=2	k=3	k=4	

These results obtained from the solution of the problem in example 1, using the trapezoidal quadrature method, are shown in Figure 1. In cases a, b, c and d, diagrams related to equation (12) are respectively plotted for m=1, m=2, m=3 and m=4.
In diagram (a) for which m =1, the difference between the errors of the trapezoidal and the modified trapezoidal quadrature methods is high, but the difference between the errors of the first and the second cases of the modified trapezoidal quadrature method are relatively low.

Diagram (b) for which m=2, is similar to the first case, with this difference that the error between the first and second cases of the modified trapezoidal quadrature method is minimal.

In diagram (c) for which m=3, the modified trapezoidal quadrature method is more accurate than the trapezoidal quadrature method, in which the first case is more accurate.

In diagram (d) for which m=4, the error between the first case of the modified trapezoidal quadrature and trapezoidal quadrature methods is very small, while the error between the second case of the modified trapezoidal quadrature and trapezoid quadrature methods is very much.

Example 2
The presented algorithm is applied for solving linear Volterra integral equation of the second type

$$u(x) = 2x - (1 - e^{1-x^2}) + \int_{-1}^x e^{-x^2+t^2} u(t)\,dt \qquad -1 \le x \le 1 \qquad (15)$$

The exact solution to this equation is $u(x) = 2x$. Tables 4 and 5 show absolute errors of numerical results calculated according to the presented algorithm, for the two cases described above. The results after Romberg extrapolation are again more accurate than results obtained using the integration formula over more points.

And here, These results obtained from the solution of the problem in example 2, using the trapezoidal quadrature method, are shown in Figure 2. In cases a, b, c and d, diagrams related to equation (12) are respectively plotted for m=1, m=2, m=3 and m=4.

In diagram (a) for which m =1, the error of the modified trapezoidal quadrature method is less than the trapezoidal quadrature method, and the error difference between the first case and the second case of the modified trapezoidal quadrature method is to an extent that it can be said that the two curves coincide.

In diagram (b) for which m =2, the modified trapezoidal quadrature method, in both cases, is more accurate than the trapezoidal quadrature method.

In diagram (c) for which m =3, the difference between the errors of the modified trapezoidal quadrature method and trapezoidal quadrature method is low, while the difference between the second case of the trapezoidal quadrature method and the trapezoidal quadrature method is very much.

In diagram (d) for which m =4, the difference between the errors of the first case of modified trapezoidal quadrature method and the trapezoidal quadrature method is very low, while the errors between the second case of the modified trapezoidal quadrature method and the trapezoidal quadrature method is very much.

Example 3

The presented algorithm is applied for solving linear Volterra integral equation of the second type

$$u(x)=e^{-x^2}-\frac{1}{2}\left(\frac{1}{e}-e^{-x^2}\right)x+\int_{-1}^{x}x\,t\,u(t)\,dt \qquad -1\le x\le 1 \qquad (16)$$

The exact solution of this equation is $u(x)=e^{-x^2}$. Tables 6 and 7 show absolute errors of numerical results calculated according to the presented algorithm, for the two cases expressed. The results after Romberg extrapolation are again more accurate than after quadrature integration over more points.

Also here, These results obtained from the solution of the problem in example 3, using the trapezoidal quadrature method, are shown in Figure 3. In cases a, b, c and d, diagrams related to equation (12) are respectively plotted for m=1, m=2, m=3 and m=4.

In diagram (a) for which m=1, the error of the modified trapezoidal quadrature method is very small compared to the trapezoidal quadrature method. And the difference error between the first case and the second case of the modified trapezoidal quadrature method is to an extent that it can be said the two curves coincide.

In diagram (b) for which m=2, the modified trapezoidal quadrature method according to the first case, is more accurate than the second case and the trapezoidal quadrature method.

In diagram (c) for which m=3, the accuracy of the modified trapezoidal quadrature method in both cases is more than the trapezoidal quadrature method.

In diagram (d) for which m=4, the difference between the errors of the first case of the modified trapezoidal quadrature method and trapezoidal quadrature method is very low, while the error between the second case of the modified trapezoidal quadrature method and the trapezoidal quadrature method is very much.

Table 4. Absolute errors of the numerical solution of (15) at x=1 with h=0.25 in the first case

m=1	m=2	m=3	m=4	m=5
$2.01*10^{-3}$	$4.94*10^{-4}$	$2.17*10^{-6}$	$1.74*10^{-9}$	$3.94*10^{-13}$
$1.32*10^{-4}$	$3.29*10^{-5}$	$3.56*10^{-8}$	$7.20*10^{-12}$	
$8.36*10^{-6}$	$2.09*10^{-6}$	$5.63*10^{-10}$		
$5.24*10^{-7}$	$1.31*10^{-7}$			
$3.28*10^{-8}$				
K=1	K=2	K=3	K=4	

Table 5. Absolute errors of the numerical solution of (15) at x=1 with h=0.25 in the second case

m=1	m=2	m=3	m=4	m=5
$2.14*10^{-3}$	$5.21*10^{-4}$	$3.45*10^{-6}$	$3.36*10^{-8}$	$8.66*10^{-10}$
$1.45*10^{-4}$	$3.58*10^{-5}$	$8.69*10^{-8}$	$9.39*10^{-10}$	
$9.31*10^{-6}$	$2.32*10^{-6}$	$2.34*10^{-9}$		
$5.90*10^{-7}$	$1.47*10^{-7}$			
$3.70*10^{-8}$				
K=1	K=2	K=3	K=4	

Table 6. Absolute errors of the numerical solution of (16) at x=1 with h=0.25 in the first case

m=1	m=2	m=3	m=4	m=5
$8.05*10^{-6}$	$2.02*10^{-6}$	$1.40*10^{-9}$	$2.66*10^{-12}$	$4.99*10^{-16}$
$4.99*10^{-7}$	$1.25*10^{-7}$	$2.45*10^{-11}$	$9.93*10^{-15}$	
$3.11*10^{-8}$	$7.78*10^{-9}$	$3.93*10^{-13}$		
$1.94*10^{-9}$	$4.86*10^{-9}$			
$1.21*10^{-10}$				
K=1	K=2	K=3	K=4	

Table 7. Absolute errors of the numerical solution of (16) at x=1 with h=0.25 in the second case

m=1	m=2	m=3	m=4	m=5
$4.79*10^{-5}$	$8.82*10^{-6}$	$4.88*10^{-9}$	$3.85*10^{-10}$	$1.12*10^{-11}$
$1.86*10^{-5}$	$5.56*10^{-7}$	$4.55*10^{-10}$	$1.27*10^{-11}$	
$5.07*10^{-6}$	$3.52*10^{-8}$	$1.96*10^{-11}$		
$1.29*10^{-6}$	$2.22*10^{-9}$			
$3.25*10^{-7}$				
K=1	K=2	K=3	K=4	

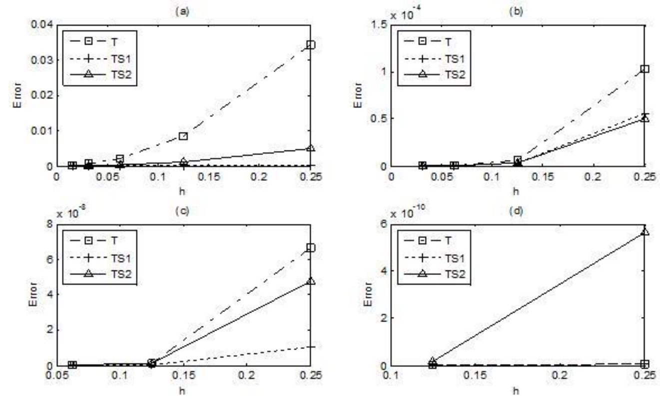

Figure 1. Absolute errors of the numerical solution of Equation (13) at x=2 for both cases of the modified trapezoidal (TS1 as the first and TS2 as the second case) and also the trapezoidal quadrature methods (T) at m=1, 2, 3 and 4 versus length step *h*

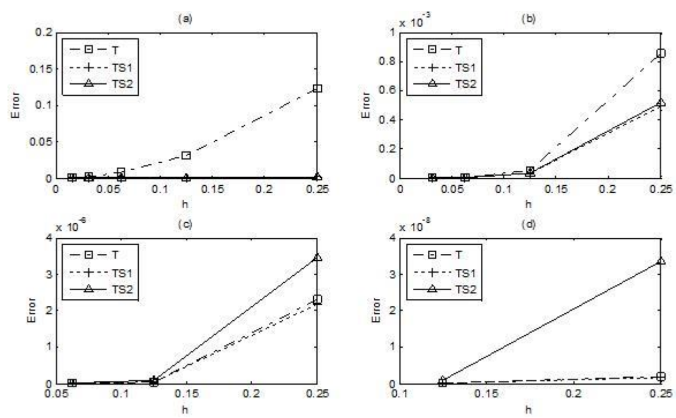

Figure 2. Absolute errors of the numerical solution of Equation (15) at x=1 for both cases of the modified trapezoidal (TS1 as the first and TS2 as the second case) and also the trapezoidal quadrature methods (T) at m=1, 2, 3 and 4 versus length step *h*

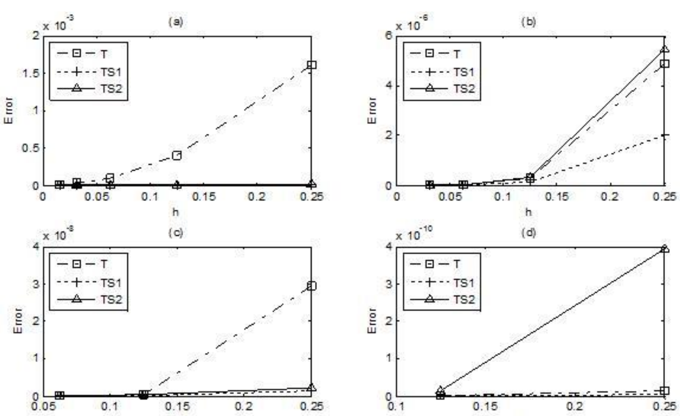

Figure 3. Absolute errors of the numerical solution of Equation (16) at x=1 for both cases of the modified trapezoidal (TS1 as the first and TS2 as the second case) and also the trapezoidal quadrature methods (T) at m=1, 2, 3 and 4 versus length step *h*

CONCULSION

Volterra integral equation of the second type is solved using the modified trapezoidal quadrature method, which in tables 2 - 6 and diagrams in the (a) sections of figures 1, 2, 3 for m=1 indicate that this quadrature method is more accurate than the trapezoidal method used in (Mesˇtrovic and Ocvirk, 2007).

With the implementation of Romberg extrapolation on modified trapezoidal quadrature method for solving the Volterra integral equation of the second type we noted that despite the discretization over a few points, more efficient and accurate result are obtained, but in contrast with the results obtained in (Mesˇtrovic and Ocvirk, 2007), the Romberg extrapolation is not as efficient when used in the modified trapezoidal quadrature method for solving the Volterra integral equation of the second type.

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